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ROLE OF CHROMATIC NUMBERS IN GRAPH THEORY

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ABSTRACT

The graph theoretic parameter that has received most attention over the years is the chromatic number and its prominence in graph theory is undoubtedly due to its involvement with the four color problem. In this paper the b-chromatic number of power graphs of complete binary trees and complete k-ary trees are discussed. The semi-strong chromatic number of various graphs is also discussed.

KEYWORDS: Power graphs, b-chromatic number, K-ary tree, semi-strong chromatic number.

INTRODUCTION

Graphs without loops or multiple edges are considered. Let G be a graph with a vertex set V and an edge set E. The degree of the vertex x in G is denoted by d(x), and the distance between two vertices x and y in G by $distG(x,y)$. The *p-th power graph* G_p is a graph obtained from G by adding an edge between every pair of vertices at distance p or less, with $p \geq 1$. It is easy to see that $G' = G$. In the literature, power graphs of several classes have been investigated **[1, 2, 3].** A *k-coloring* of G is defined as a function c on $V(G) = \{v_1, v_2 \text{------}, v_n\}$ into a set of colors $C = \{1, 2\ldots.., k\}$ such that for each vertex v_i, with $1 \leq i \leq n$, $c_{vi} \in C$. A *proper k-coloring* is a k-coloring satisfying the condition $c_x \neq c_y$ for each pair of adjacent vertices $x,y \in V(G)$. A *dominating proper k-coloring* is a proper k-coloring satisfying the following property P: for each i, $1 \leq i \leq k$, there exists a vertex x_i of color i such that, for each j, with $1 \leq j \neq i \leq k$, there exists a vertex y_j of color j adjacent to x_i. A set of vertices satisfying the property P is called a *dominating system*. Each vertex of a dominating system is called a *dominating vertex*. The *b-chromatic number* of a graph G is defined as the maximum k such that G admits a dominating proper k-coloring. The b-chromatic number was introduced in **[4].** The motivation for *achromatic number* **[5, 6]),** comes from algorithmic graph theory. The achromatic number of a graph G is the largest number of colors which can be assigned to the vertices of G such that the coloring is proper and every pair of distinct colors appears on an edge. A proper coloring of a graph G using $k > \chi(G)$ colors could be improved if the vertices of two color classes could be recolored by a single color so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by the achromatic number. A more versatile form of recoloring strategy would be to allow the vertices of a single color class to be redistributed among the colors of the remaining classes, so as to obtain a proper coloring. The semi-strong chromatic number $x_s(G)$ of a graph G is the minimum order of a partition L of $V(G)$ such that every set S in L has the property: no vertex of G has two neighbors in S. The number $x_s(G)$ is determined for various known graphs including trees and block graphs, and some bounds are obtained for it.

CONCEPTS AND NOTATIONS

A graph G = (V, E) consists of a set denoted by V and a collection E of unordered pairs of distinct elements of V. Each element of V is called a vertex or a point or node. The element of E is called an edge or a line or a link denoted as e. The unordered pair {x,y} is an edge in G if and only if {x,y} = xy lies in E. The vertices x and y are called adjacent vertices if and only if xy is an edge in G. Also x and y are end vertices of an edge in G. If e is an edge in G then e is incident with its vertex. A graph G is complete if and only if any two vertices are adjacent. $N_G(x)$ is open neighborhood of x in G: $y \in V$: x, $y \in E\}$. $N_G[x]$ is closed neighborhood of x in G : $N_G(x) \cup \{x\}$. $d_G(x)$ is the degree of x in G: the number of edges incident with x in G. $d_G(S)$ is the degree of S in G : Minimum of the $d_G(x)$ x in S. A walk of length k is a finite sequence $W = x_0, x_1, x_2 \ldots x_k$ of vertices such that any two consecutive elements of W form an edge in G. If all edges of W are distinct, then W is called a trail. If $x_0 = x_k$ then W is a closed walk. A closed trail is called a circuit. If the vertices of a walk are distinct then W is a path. If in a walk $W = x_0, x_1, x_2 \ldots x_k$, $x_0 = x_k$ and $x_1, x_2, ..x_{k-1}$ are all distinct then the walk W is called a cycle of length k or a k-cycle. If $x = x_0$ and $y = x_k$ of a walk W =

$x_0, x_1, x_2 \ldots x_k$ then W is called x-y walk of length k. A graph G is connected if for every pair x,y of vertices there exists a x - y path. Otherwise G is disconnected. A tree is an acyclic connected graph .A binary tree is a rooted plane tree where each vertex has at most two children. A k-ary tree is a rooted tree where each vertex has at most k children. $dist_G(x, y)$ is the minimum length of a x - y walk. The eccentricity of x, ecc(x), is max $\{dist_G(x,y): y \in V\}$.The radius of G, rad(G), is $\min\{ecc(x): x \in V\}$.The diameter of G, diam(G), is $\max\{ecc(x): x \in V\}$.The distance of x, $dist_G(x)$, is $\sum_{y \in V} d(x, y)$.

Let G = (V, E) be a graph and let S be a subset of V. The induced sub graph < S > of G is the maximal sub graph of G with point set S. That is two points of S are adjacent in < S > if and only if they are adjacent in G. A k-coloring of G is defined as a function c on V (G) = $\{x_1, x_2, x_3 \ldots x_n\}$ to a set C = {1, 2, 3…k} of colors such that for each vertex x_i, with $1 \leq I \leq n$, we have $c_{xi} \in C$. A proper k-coloring of G is a k-coloring satisfying the condition $c_x \neq c_y$ for each pair of adjacent vertices x,y in V. The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of G. A graph G is k-colorable if $\chi(G) \leq k$. A stable set or an Independent set in a simple graph G is a set of pair wise non-adjacent vertices in G. The independence number α (G) is the maximum size of an independent set in G. In fact α (G) = max {| J|: J is a stable set in G}. A graph G = (V, E) is bipartite if V is the union of two disjoint independent sets in G and k-partite if V is the union of k disjoint independent sets in G. A set S of vertices of G is a dominating set of G if every vertex of G is adjacent to at least one vertex in S. The set of vertices having the same color is called the color class. The edge-chromatic number. Graph G is k-chromatic if $\chi(G) = k$. A graph G with no isolated vertices is color-critical if and only if χ (G-e) < χ (G) for every e Є E (G). Let S be a collection of subsets of a finite set X. The smallest subset Y of X that meets every member of S is called the vertex cover or hitting set. A maximal independent set is therefore an independent set containing the largest possible number of vertices.

THE POWER OF A GRAPH

The k^{th} *power* of a graph *G* is the graph G^k whose vertex set is the same as that of *G* and whose edge set consists of pairs of vertices (x,*y*) whenever vertices x and y are distance-*k* neighbors in *G*. k^{th} power of graph can also be defined as follows: Let G be a graph and $k \geq 1$, the k^{th} power graph G^k is a graph obtained from G by adding an edge between every pair of vertices at a distance p or less. For example, the graph G^2 and G^3 are referred as the square and cube, respectively of graph G. A graph with its square and cube are shown in figure 1.

Proposition 3.1: *for any graph G, $G^1 = G$.*
If two vertices are at a distance 1 then they are already adjacent in G. This proves the proposition.

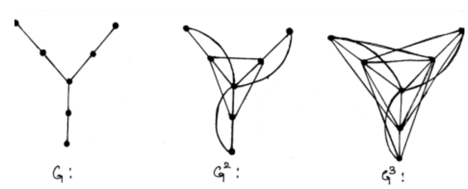

Fig 1: The power graphs of G

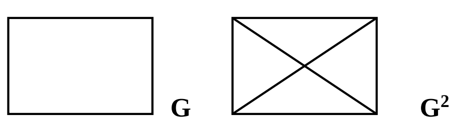

Fig 2: The power graph of complete graph

Some illustrations showing that the square of a graph is a complete graph are shown in figure 2.
The following illustration in figure 3 shows that the square is not complete where as the cube is complete.

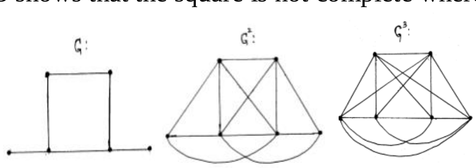

Fig 3: Square and cube graphs of G

Proposition 3.2: For any graph G of order n, if diam(G) ≤ p, then G^p is a complete graph.
Let diam(G) ≤ p. Then any two vertices are at a distance ≤ p. So, any two vertices of G are adjacent in G^p. Therefore G^p is a complete graph.

B-CHROMATIC NUMBER

The b-chromatic number of *G* is defined as the maximum number *k* of colors that can be used to color the vertices of *G*, such that a proper coloring can be obtained and each color *i*, with $1 \le i \le k$, has at least one representant x_i adjacent to a vertex of every color *j*, $1 \le j \ne i \le k$, the exact value for the b-chromatic number of power graphs of a path is given and bounds for the b-chromatic number of power graphs of a cycle are determined. A *k-coloring* of *G* is defined as a function *c* on $V(G) = \{v_1, v_2, \ldots, v_n\}$ into a set of colors $C = \{1,2,\ldots,k\}$ such that for each vertex *vi*, with $1 \le i \le n$, we have $c_{vi} \in C$. A *proper k-coloring* is a *k* coloring satisfying the condition $C_x \ne C_y$ for each pair of adjacent vertices $x,y \in V(G)$. A *dominating proper k-coloring* is a proper *k*-coloring satisfying the following property *P*: for each *i*, $1 \le i \le k$, there exists a vertex x_i of color *i* such that, for each *j*, with $1 \le j \ne i \le k$, there exists a vertex y_j of color *j* adjacent to x_i. A set of vertices satisfying the property *P* is called a *dominating system*. Each vertex of a dominating system is called a *dominating vertex*. The *b-chromatic number* j(*G*) of a graph *G* is defined as the maximum *k* such that *G* admits a dominating proper *k*-coloring. The motivation, similarly as for the previously studied *achromatic number* comes from algorithmic graph theory. The achromatic number y(*G*) of a graph *G* is the largest number of colors which can be assigned to the vertices of *G* such that the coloring is proper and every pair of distinct colors appears on an edge. A proper coloring of a graph *G* using *k* >c(*G*) colors could be improved if the vertices of two color classes could be recolored by a single color so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by the achromatic number. A more versatile form of recoloring strategy would be to allow the vertices of a single color class to be redistributed among the colors of the remaining classes, so as to obtain a proper coloring. Table 1 shows the b-chromatic numbers of certain standard graphs.

Graph G	**φ(G)**
Kp	P
$\overline{K_p}$	0
K_p-x	p-1
$K_{m,n}$	2
P_n	2
C_{2n}	2
C_{2n+1}	3

Table 1: b-chromatic number of standard graphs

Proposition 4.1: Assuming that the vertices $x_1, x_2, x_3, \ldots, x_n$ of G are ordered such that $d(x_1) \ge d(x_2) \ge, \ldots, \ge d(x_n)$. Then $\varphi(G) \le m(G) \le \Delta(G) + 1$ where $m(G) = \{i: 1 \le i \le n, d(x_i) \ge i-1\}$
R. W. Irving and D. F. Manlove proved that finding the b-chromatic number of any graph is a NP-hard problem, and they gave a polynomial-time algorithm for finding the b-chromatic number of trees. Kouider and Mah´eo gave some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. They gave, in particular, a lower bound for the b-chromatic number of the cartesian product of two graphs where each one has a stable dominating system. More recently it was characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper *k*-coloring even for connected bipartite graphs and *k*=Δ(G)+1. They also determine the asymptotic behavior for the b-chromatic number of random graphs.
Some of the observations are made as below.

Theorem 4.2: For any graph G, $\varphi(G) \le \chi(G)$.
Proof: A proper coloring of a graph G using $k > \chi(G)$, colors could be improved if the vertices of two color classes could be re-colored by a single color so as to obtain a proper coloring This form of re-coloring Strategy would be to allow the vertices of a single color class to be redistributed among the colors of the remaining classes so as to obtain a proper coloring. The largest number of colors for which such a re-coloring strategy is not possible is φ(G). Thus

$\varphi(G) \le \chi(G)$. This proves the theorem. The above theorem shows that the results and bounds for the b-chromatic numbers are interesting. The following theorem due to Irving et al. (1999) gives the bounds for b-chromatic numbers.

THE POWER GRAPHS OF BINARY TREES

T_h denotes a complete binary tree of height h and T_h^p denotes the pth power graph of T_h.

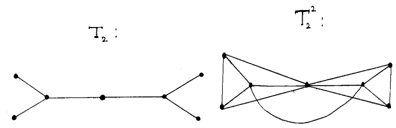

Fig 4: The power graph of complete binary tree

These properties are needed to find the b-chromatic numbers of the power graphs of binary trees and k-ary trees. The vertices of T_h are denoted by $x_1, x_2, \ldots x_{2^{h+1}-1}$ from level 0 to level h. we say that a vertex x belongs to the level denoted by L_x = dist $T_h(x_1,x)$. Let di be the degree of any vertex on level i in $T_h^{\ p}$.

Lemma 5.1: For $\frac{p}{2}$ < h< 2p, the order of degrees of T_h^p, with $p \ge 2$, is given by:

1. for $\frac{p}{2}$ < h< p, $d_0 = d_1 = \ldots d_{p-h} > d_{p-h+1} \ge d_{p-h+2} \ge \ldots \ge d_h$.
2. for $p \le h \le 2p$, $d_{h-p} \ge d_{h-p-1} \ge \ldots \ge d_1 \ge d_0 \ge d_{h-p+1} \ge d_{h-p+2} \ge \ldots \ge d_h$.

Proof: For each $v \in E(T_h)$, let T(v) be the sub tree of T_h rooted on v. Let x and y be two vertices of T_h such that $L_y = L_x + 1$. From the structure of the tree, we observe that if $\frac{p}{2}$ <h< p, all vertices of levels 0 to p-h are adjacent to all other vertices. So, $d_0 = d_1 = \ldots = d_{p-h}$. f $p \le h \le 2p$ and $L_y \le h-p$, then x and y have the same number of neighbours respectively T(x) and T(y). More over as $L_x > L_y$, we observe that the number of remaining neighbours of y is larger than that of x. So, $d_{h-p} \ge d_{h-p-1} \ge \ldots \ge d_1 > d_0$.In the same way, if $L_y \ge |h-p| + 2$, we observe that y has less neighbours in T(y) than x in T(x). And the number of remaining neighbours of y is less or equal to that of x. So,
$d_{|h-p|+1} \ge d_{|h-p|+2} \ge \ldots \ge d_h$.

Finally, $d_{h-p+1} = \; 2^{p-1} - 2 + \sum_{j=0}^{h-p} 2^{p-(h-p+1-j)}$

$$= 3*2^{p-1} - 2 - 2^{2p-h-1}$$

As $d_0 = 2^{p+1} - 2$, we deduce that $d_0 > d_{h-p+1}$. This proves the lemma. The following figure shows the order of degrees of T_3^p for p = 2 and p = 4.

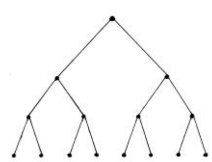

Fig 5: Order of degree of T_3^p

Degrees	h = 3, p = 2	h = 3, p = 4
d_0	6	14
d_1	8	14
d_2	5	10
d_3	3	8

Table 2: Degrees of vertices of T_h^p

The following lemma proves that there exists a level M in T_h^p satisfying $2^{m+1} \geq d_m$.

Lemma 5.2: There exists at least one level M in T_h^p, with p≥2, and $0 \leq M \leq h$, such that $2^{M+1} \geq d_M$.

Proof: We give the proof by the method of contradiction. Suppose that $2^{i+1} < d_i$ with $0 \leq i \leq h$. Since there are $2^{h+1} - 1$ vertices in T_h, we have $d_i \leq n-1 = 2^{h+1} - 2$. In particular, we have $2^{h+1} < d_h \leq 2^{h+1} - 2$, a contradiction. This proves the lemma.
The following lemma gives a lower bound for the level M.

Lemma 5.3: For $\frac{p}{2} < h < 2p$, we have $M \geq |h-p|+1$ for each $p \geq 2$.

Proof: We give the proof by the method of contradiction. Suppose that there exists a level M′, with $0 \leq M' \leq |h-p|$, verifying $d_{M'} \leq 2^{M'+1}$. There are two cases.

Case 1: $\frac{p}{2} < h < p$. As $d_0 = 2^{h+1}-2$, by Lemma 5.1 we have $d_0 = d_1 = \ldots = d_{p-h} = 2^{h+1}-2$. More over, as p/2 < h, we have $2^{p-h+1} \leq 2^h$. As $M' \leq p - h$, $2^{h+1} - 2 = d_{M'} \leq 2^{M'+1} \leq 2^{p-h+1} \leq 2^h$. This is a contradiction.
Case 2: $p \leq h \leq 2p$. It is easy to see that for M′ =0, there is a contradiction ($d_{M'} = d_0 = 2^{p+1}-2$ and $2^{M'+1} = 2$, with p≥2). As p≥2, we have $d_1 \geq d_0 + 3$. Then Lemma 5.1 shows that $d_{M'} \geq d1 \geq 2^{p+1}+1$, for $1 \leq M' \leq h-p$. More over as $h \leq 2p$, we have $2^{h-p+1} \leq 2^{p+1}$. Then, for each level $1 \leq M' \leq h-p$, we have, $d_{M'} \geq 2^{p+1} \geq 2^{h-p+1} > 2^{M'+1}$ that is a contradiction. Therefore there does not exist a level M′, with $0 \leq M' \leq h - p$, verifying $d_{M'} \leq 2^{M'+1}$. This proves the lemma. The following lemma proves that vertices $x_1, \ldots x_k$ belong to levels 0…p.

Lemma 5.4: For $\frac{p}{2} < h \leq 2p$ and $k = \max\{2^M - 1, d_M + 1\}$, we have $L_{xk} \leq p$.

Proof: We prove this by the method of contradiction. Suppose $L_{xk} > p$. If $\frac{p}{2} < h < p$, it is easy to see that $L_{xk} \leq h < p$, a contradiction. For $p \leq h \leq 2p$, we have two cases. Firstly if $k = 2^M - 1$, the vertex x_k is the last vertex of the level M-1, So L_{xk} = m-1. As $L_{xk} > p$, we have $M \geq p+2$. By its definition, M is the first level verifying $d_M \leq 2_{M+1}$. As $M \geq p+2$, we have $d_{p+1} > 2^{p+2}$. However $h \leq 2p$, then $h-p+1 \leq p+1$ and lemma 5.1 proves that $d_{p+1} \geq d_{h-p+1} < d_0 = 2^{p+1}-2$, a contradiction. Secondly, $k = d_M + 1$. As $L_{xk} > p$, then $k \geq 2^{p+1}$and $d_M \geq 2^{p+1} - 1$. lemma 5.3 proves that $M \geq h-p+1$ and Lemma 5.1shows that $d_M < d_0 = 2^{p+1}-2$. then, we deduce that $2^{p+1}-2 = d_0 > d_M > 2^{p+1}-1$, a contradiction. This proves the lemma. In the following two lemmas, we give the bounds for degrees of vertices of T_h^p.

Lemma 5.5: For $\frac{p}{2} < h \leq 2p$ and $k = \max\{2^M - 1, d_M + 1\}$, then for each vertex x_i, with $1 \leq i \leq k$ we have $d(x_i) \geq k-1$.
Proof: The lemma 5.3 gives $M \geq |h-p|+1$.
Case 1: $k = d_M + 1$. Then each vertex x_i, $1 \leq i \leq k$, is on a level $L_{xi} \leq M$. More over, Lemma 5.1proves that $d_{|h-p|} \geq d_{|h-p|-1} \geq \ldots \geq d_0 > d_{|h-p|} + 1 \geq d_M = k-1$. So $d(x_i) \geq k-1$, with $1 \leq i \leq k$.
Case 2: $k = 2^M - 1$.Here $L_{xk} =$ M-1 and lemma5.4 shows that $L_{xk} \leq p$, so $M \leq p+1$. If $M = |h-p| + 1$, then we have two cases. First if $\frac{p}{2} < h < p$, Lemma 5.1shows that $d_0 = d_1 = \ldots = d_{M-1} = 2^{h+1}$ .As $M \leq h$, we have $d_i \geq 2^{h+1} - 2 \geq 2^{M+1} - 2 > k$. Second if $p \leq h \leq 2p$, Lemma 5.1shows that $\min\{d_i : 0 \leq i \leq M-1\} = d_0 = 2^{p+1} - 2$.As $M \leq p+1$, we have $d_i \geq 2^{p+1} - 2 \geq 2^M - 2 = k-1$. Finally, if $M > |h-p| + 1$, Lemma 5.1proves that $\min\{d_i: 0 \leq i \leq M-1\} = d_{M-1}$. As M is the first level verifying $d_M \leq 2^{M+1}$, we have $dM_{-1} > 2^M$. So $d(x_i) \geq k-1$, with $1 \leq i \leq k$. This proves the lemma.

Lemma 5.6: For $\frac{p}{2} < h \leq 2p$ and $k = \max\{2^M - 1, d_M + 1\}$, then for each vertex x_i, with $k+1 \leq i \leq 2^{h+1} - 1$ we have $d(x_i) \leq k-1$.
Proof: lemma 5.3 proves that $M \geq |h-p| + 1$. If $k = d_M + 1$, then $d_M < k$ and Lemma 5.1shows that $\max\{d_i\, M \leq I \leq h\} = d_M < k$. If $k = 2^M - 1$, then $k \geq d_M + 1$. So $d_M < k$ and lemma 5.1 gives $\max\{d_i: M \leq i \leq h\} = d_M < k$. This proves the lemma.

The following lemma shows that if a graph G has dominating proper k-coloring, then all added vertices with degree less than k can be colored the keep a proper k-coloring of G.

Lemma 5.7: Let G′ be an induced sub graph of G by V′ ⊆V. If G′ admits a dominating proper k′- coloring and each vertex of G\G′ has a degree in G less than k′, then G admits a dominating proper k-coloring with the same dominating system.

Proof: Let x be a vertex of G/G′ with degree d(x). Let A be the set of adjacent colors to x in G′. If V = V′ (that is V (G/G′) =ϕ) then the results holds. Otherwise, as |A| ≤ d(x) < k′, then there is at least one color c not adjacent to x, belonging to {1,2,…,k′}. Then we put c_x = c. Let now G′ be an induced sub graph given by V′∪ {x}. The graph G′ verifies the hypothesis of Lemma. So, we repeat this process for each vertex of G/G′ until V′=V.

B-CHROMATIC NUMBER OF K-ARY TREES

Kouider and etal gave some lower and upper bounds for the b-chromatic number of Cartesian product of two graphs. They gave, in particulars, a lower bound for the b-chromatic number of Cartesian product of two graphs where each one has a stable dominating system. More recently Brice Effantin and Hamamache Kheddouci characterized bipartite graphs for which the lowerbound on the b-chromatic number is attained and proved the NP- completeness of the problem to decide whether there is a dominating proper k-coloring ever for connected bipartite graphs and k = Δ(G) + 1. They also determine the asymptotic behavior for the b-chromatic number of random graphs. The following theorem finds the b-chromatic number of power graphs of complete binary trees.

Theorem 6.1: Let T_h be a complete binary tree of height h. The b-chromatic number of T_h^p, with p ≥ 2, is:

$$\varphi(T_h^p) = \begin{cases} 2^{h+1}-1 & if \quad 2h \le p) \\ \max\{2^M-1, d_M+1\} & if \quad p < 2h \le 4p \\ 3(2^p-1)+1 & if \quad h \ge 2p+1 \end{cases}$$

where M is the first level verifying $2^{M+1} \geq d_M$.

Proof: Let p = 1. Then for h = 1, 2 and 3 $\varphi(T_h^p)$ = h + 1. Otherwise $\varphi(T_h^p)$ = 4.

So we assume that p ≥ 2.

Result (a): $\varphi(T_h^p) = 2^{h+1} - 1$ if h ≤ p/2

Suppose h ≤ $\frac{p}{2}$. One can see that the graph T_h^p, with p ≥ 2, is a complete graph. Therefore, for any graph G of order n,if diam(G) ≤ p then φ(G) = n with p ≤ 2, $\varphi(T_h^p) = 2^{h+1} - 1$.

Result (b): $\varphi(T_h^p)$ = max { 2^M –1, d_M+1} if $\frac{p}{2}$ < h ≤ 2p.

Firstly we prove that $\varphi(T_h^p)$ ≥ max {2^M –1, d_M+1}. Lemma5.2 proves that there exists a level M verifying $2^{M+1} \geq d_M$.

Let k= max {2M –1, d_M+1}. We give a proper k-coloring for T_h^p in three steps.

Step1: We color the k first vertices with k different colors as in the following figure.

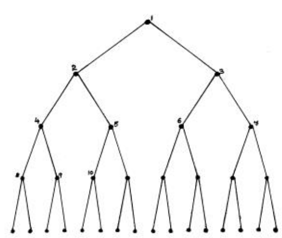

Fig 6: Coloring of complete tree

Step 2: For each m=1, 2, 3,.., k, we color some neighbors of x_m to become it a dominating vertex. Let $N^l(x_m)$ be the set of neighbors of x_m on level l, with $0 \le l \le h$. For each color j, with $1 \le j \ne c_{xm} \le k$ and x_m is not adjacent to j, we put j on a non colored neighbor of x_m such that a vertex of $N^{l+1}(x_m)$ will be colored if all vertices on $N^l(x_m)$ are colored, with $0 \le l \le h-1$ and $1 \le m \le k$.

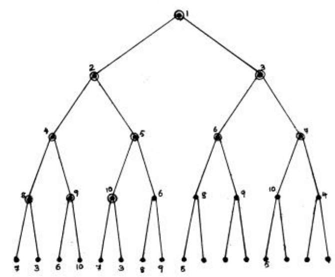

Fig 7: Coloring of complete tree

Now X be the set of vertices $\{x_1,x_2,x_3,\ldots,x_k\}$. Next we prove by induction on m that the coloring given by step 2 from x_1 to x_m is proper, with $1 \le m \le k$. lemma5.4proves that $L_{xk} \le p$. Then x_i is adjacent to each vertex of X and by construction all vertices of X have different colors. So, for m = 1, the induction hypothesis is verified. Suppose that the coloring given by step2 is proper for x_1 to x_m, with $1 \le m < k$. Then we prove this hypothesis for m + 1. Let v be a non colored neighbor of x_{m+1} on T_h^p . Let y be colored vertex on T_h^p such that $(v,y) \in E(T_h^p)$. We denote by P[v,y] the path with end vertices v and y in T_h. By applying Step2 for x_{m+1}, we color v and we prove that $cv \ne cy$ by construction. Suppose $c_v = c_y$ .By construction, as y is already colored, $L_y \le L_v$. Moreover we have $(y,x_{m+1}) \notin E(T_h^p)$ since cv=cy and by construction of Step2, each vertex is not adjacent to the same color. Let T ′ the sub tree rooted on x_{m+1}. If $\{v,y\} \in T'$, as $L_y \le L_v$ and $(v,x_{m+1}) \in E(T_h^p)$ then $(y,x_{m+1}) \in E(T_h^p)$, a contradiction. If only $v \in T'$,(or only $y \in T'$) , then $x_{m+1} \in P[v,y]$ since x_{m+1} is the root of T ′. As $(v,y) \in E(T_h^p)$, we deduce $(y, x_{m+1}) \in E(T_h^p)$,, a contradiction. If $\{v,y\} \notin T'$, let T_a be the sub tree rooted on $x_a=\max\{x_i:1\le i\le m\}$ and $\{v,x_{m+1}\} \in T_a$. If $y \in T_a$, as Ly ≤ Lv and (v,x_{m+1}), let T_b be the sub tree rooted on x_b such that $x_b= \max\{x_i : 1\le i\le m\}$ and $\{y,x_{m+1}\} \in T_b$. One can note that $L_{xb} < L_{xa}$. Then, $dist_{Th}(x_{m+1},y) = dist_{Th}(x_{m+1},x_a) + dist_{Th}(x_a, x_b) + dist_{Th}(x_b,y)$ ………(1) More over as $(v,y) \in E(T_h^p)$, $dist_{Th}(v,y) = dist_{Th}(v,x_a) + dist_{Th}(x_a, x_b) + dist_{Th}(x_b,y) \le p$ …………(2) By constructuion $L_{xm+1} \le Lv$, so $dist_{Th}(x_{m+1},x_a) \le dist_{Th}(v,x_a)$.Then from (1) and (2) we deduce that $dist_{Th}(x_{m+1},y) \le dist_{Th}(v,y) \le p$ which is a contradiction since $(x_{m+1}, y) \notin E(T_h^p)$. Thus the coloring by Step 2 for x_m with $1\le m \le k$ is proper.

Step 3: By lemma5.5, each vertex x_i, with $k+1\le i\le 2^{h+1}-1$, verifies $d(x_i) < k$. Thus lemma5.6 allows us to extend the coloring given by Step 1 and Step 2 to the remaining vertices to have a proper coloring.

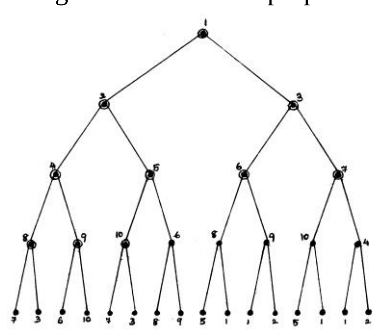

Fig 8: Coloring of complete tree

Now the three steps show that the coloring is proper. Lemma5.5 proves that each vertex x_i, with $1\le i \le k$, verifies $d(x_i) \ge k-1$. Moreover by applying Step 2, each vertex x_m, with $1\le m\le k$, is adjacent to each color j, $1\le j \ne c_m \le k$. Thus this coloring is a dominating proper k-coloring where the dominating system is X. This proves that $\varphi(T_h^p) \ge \max\{ 2^M -$

1, d_M+1}.Next we prove that $\varphi(T_h^p) \leq \max\{2^M-1, d_M+1\}$. The proof is by contradiction. Suppose that there exists a dominating proper k ′-coloring such that $k' > \max\{2^M-1, d_M+1\}$. There are two cases.

Case 1: $2^M-1 \geq d_M+1$.
Then by lemma 5.3, $k' \geq 2^M \geq 2^{|h-p|}+1$. There is at least one dominating vertex x on level L_x such that $M \leq L_x \leq h$. To be dominating vertex, x must have a degree greater than k ′- 1. As Lemma 5.1proves that $\max\{d_M, d_{M+1}, \ldots, d_h\} = d_M$, we have $d_M \geq d_{Lx} \geq k'-1$, $D_M+1 \geq k' \geq 2^M$. that is a contradiction.

Case 2: $d_M+1 > 2^M-1$.
Then $k' \geq d_M \geq 2^M+1$. Then, there is at least one dominating vertex x on level L_x, with $M \leq L_x \leq h$. On level M, a vertex can be adjacent to at most d_M vertices. So no vertex of level M can be a dominating vertex. Moreover, by Lemma 5.1and lemma5.3 we have $d_{|h-p|+1} \geq d_{|h-p|+2} \geq \ldots, \geq d_h$. So, no level i, with M< I ≤ h, has a vertex with degree more or equal than k ′-1, a contradiction. So, we have $\varphi(T_h^p) \leq \max\{2^M-1, d_M+1\}$.We deduce from the above discussion, $\varphi(T_h^p) = \max\{2^M-1, d_M+1\}$.

Result (c): $\varphi(T_h^p) = \Delta(T_h^p)+1 = 3(2^p-1)+1$ if $h \geq 2p+1$. Let $k = \Delta(T_h^p)+1 = 3(2^p-1)+1$. The Proposition 4.1 shows that $\varphi(T_h^p) \leq \Delta(T_h^p)+1$. Let $X_1 = \{x_\alpha : \alpha = 2^{p+1}+j;\ j=0 \text{ to } 2p-1\}$. $X_2 = \{x_\alpha : \alpha = 2^{p+1}+j;\ j=2p \text{ to } 2^{p+1}-1\}$.Let X_3 be the set of the $k-2^{p+1} = 2^p-3$ first vertices of level p. Let $X = X_1 \cup X_2 \cup X_3$.We prove by construction that $\varphi(T_h^p) \geq \Delta(T_h^p)+1$. Firstly, we put k different colors on each vertex of X.

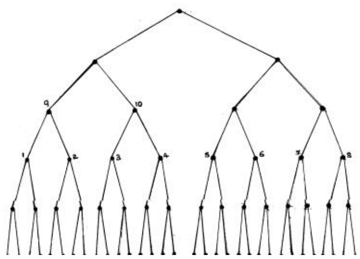

Fig 9 : coloring after first step

Secondly, we color vertices of levels 0 to p. Let V_1 (resp. V_2) be the set of non colored vertices of levels 1,2,…,p in the left(resp.right) sub tree of T_h. Let C_{x1} and C_{x2} be the sets of colors of respectively X_1 and X_2. On each vertex of V_1 (resp.V_2), we put a not used color of C_{x2}(resp.C_{x1}). Since $|C_{x1}| = |C_{x2}| = 2^p$ and $|V_1| \leq |V_2| \leq 2^p-1$, then each vertex of $V_1 \cup V_2$ has a different color as others and it remains at least one color not used in C_{x2}. So we put this color on x_1.

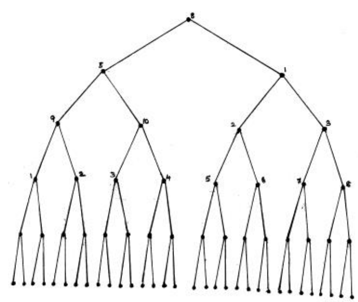

Fig 10: Coloring after second step

Thirdly, to color the remaining vertices we use the same coloring as in the second step of the Result (b). For each vertex x of X_3, we put each color j, with $1 \leq j \neq C_x \leq k$ and x is not adjacent to j, on each non colored vertex x ′, where $x' = \min\{x_i : 1 \leq I \leq 2^{h+1}-1\}$ and $(x, x') \in E(T_h^p)$,We start again this third step for each vertex of X_1 and X_2. Finally, if some vertices of T_h are not colored, we start again this step for each vertex of levels p + 2 to h - p.

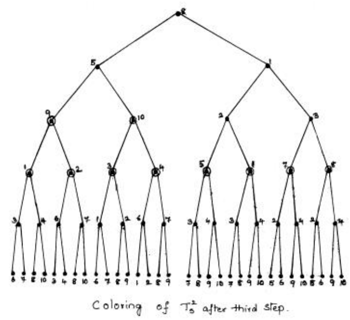

Fig 11: Coloring after third step

By construction, two neighbors will not have the same color. Moreover, as d(x) = $\Delta(T_h^p)$ for each vertex of levels p to h-p, all vertices of levels h-p+1 to h are colored. Therefore we obtain a dominating proper k-coloring where X is the dominating system. This proves that $\varphi(T_h^p) \geq \Delta(T_h^p)+1$. Therefore we deduce that $\varphi(T_h^p) = \Delta(T_h^p)+1 = 3(2^p-1)+1$. This proves the theorem. In the following theorem we find the b-chromatic number of power graph of complete k-ary trees.

Theorem 6.2: Let T_h be a complete k-ary tree of height h. The b-chromatic number of T_h^p, with p ≥ 2, is:

$$\varphi(T_h^p)=\begin{cases} \dfrac{k^{h+1}-1}{k-1} & if \quad 2h \leq p \\ \max\{\dfrac{k^{M}-1}{k-1}, d_M+1\} & if \quad p < 2h \leq 4p \\ 2+\dfrac{k^{p+1}-k}{k-1}+\dfrac{k^{p+1}+k-k^p-k^2}{(k-1)^2} & if \quad h \geq 2p+1 \end{cases}$$

where M is the first level verifying $\frac{k^{p+1}-1}{k-1} \geq d_M$

Proof: We extend different algorithms of Theorem 6.2 to the power complete k-ary tree. Indeed, if h ≤ $\frac{p}{2}$, we have a complete graph and Theorem 4.3 shows that $\varphi(T_h^p)=\frac{k^{p+1}-1}{k-1}$. For the second result p < 2h ≤ 4p, we can find a level M verifying $\frac{k^{M+1}-1}{k-1} \geq d_M$, and with the same construction as for a complete binary tree, we can color a power complete k-ary tree, with $\varphi(T_h^p)=\max\{\frac{k^{M}-1}{k-1}, d_M+1\}$ colors. For the third result, a simple modification is needed since for a complete k-ary tree, with k ≥ 3, there are more than 2 sub trees of T_h rooted in level 1. Moreover, there are more than $\Delta(T_h^p)$ vertices on level p +1, then there are no dominating vertices on the level p. However, we can prove that the number n′ of vertices from level 0 to p is less than $\Delta(T_h^p)+1$. We have n′ $= \sum_{i=0}^{p} k^i = \frac{k^{p+1}-k}{k-1}+1$.

As $\Delta(T_h^p)+1= 2+\dfrac{k^{p+1}-k}{k-1}+\dfrac{k^{p+1}+k-k^p-k^2}{(k-1)^2}$, the construction given for the third result can be done.

SEMI- STRONG CHROMATIC NUMBERS

The 'semi-strong chromatic number' $x_s(G)$ of a graph *G* is the minimum order of a partition L of *V (G)* such that every set *S* in *L* has the property : no vertex of *G* has two neighbors in *S*. The number *Xs (G)* is determined for various known graphs including trees and block graphs, and some bounds are obtained for it. Also graphs *G* for which x_s, *(G)* = I *v(G)* I are characterized and an open problem is stated. Let *G* = (*V, E*) be a graph. For u∈*V, let N (u) = {v ∈V: UV∈E}* and *N [u] = N (u) U {u}.* According to Berge1 (p.448), a set *S* ⊂ V is 'strongly stable if I *N [u]* ∩ *S*I ≤ 1 for all u ∈V. Following this definition, we call *S* 'semi-strongly stable (s-strongly stable)' if I *N (u)* ∩ *S* I ≤ *1* for all *u* ∈V. Clearly, *S* is s-strongly stable if and only if *S* contains no two neighbors of a vertex. Note that a strongly stable set is independent, but an s-strongly stable set need not be so. In fact, if S is s-strongly stable, every component in the sub graph $\langle S \rangle$ induced by S is either K_1 or K_{2s}.Some elementary observations.

Proposition 7.1:

(1) $\chi_s(K_p) = p$, $p \neq 2$, $\chi_s(K_{m,n}) = n$, $m < n$.

(2) For the cycle C_n or n vertices,

$$\chi_s(C_n) = \begin{cases} 2 \text{ if } n = 0 \bmod 4 \\ 3 \text{ otherwise} \end{cases}$$

(3) For the path P_n on n vertices,

$$\chi_s(P_n) = \begin{cases} 1 \text{ if } n \leq 2 \\ 2 \text{ if } n \geq 3 \end{cases}$$

(4) For the wheel W_n on n vertices, $\chi_s(W_n) = n$

Proposition 7.2: If $\triangle(G)$ is the maximum degree of a graph G, then, $\triangle G \leq \chi_s(G)$. (1) With equality for trees ($\neq K_1$).

Proof: Let ν be a vertex of G with deg ν = △.In any s-coloring of G, the △ neighbours of ν must be given different colors. Hence (1) follows. Suppose now G is a tree. We can s-color any vertex of G and extend to an s-coloring of all of G with △ colors: Assume $X \subset V(G)$ is s-colored. Let $\in V(G)-X$ such that $u \in N(v)$ for some $v \in N$. Since v is the only neighbour of u in X, there are △-1 distinct colors different from that of v available for $N(u)–v$.Assign a color not used in $N(v) – u$ to u.We now have an s-coloring of $X \cup N[u]$ and can continue by induction. Hence $\chi_s(G) \leq \triangle$ and equality holds in (1). This completes the proof of the proposition. We observe that for graph G, $\chi_s(G) = 2$ if and only if every component of G is either a path P_n, $n \geq 3$, *or a cycle* $C_{n,}$ $n = 0$ mod 4. This follows from Propositions 7.1 and Propositions 7.2 Note that it is possible for χ_s to be arbitrarily larger than△. We establish this fact in the sequel.

Proposition 7.3: if $\chi(G)$ is the chromatic number of a connected graph G, then, $\chi(G) \leq \chi_s(G)$ if $G \neq \chi_s\ K_2$.

Proof: It is well known that $\chi(G) \leq \triangle + 1$. Thus from (1).$\chi(G) -1 \leq \triangle(G) \leq \chi_s(G)$.If $\chi(G) -1 \leq \triangle(G)$, then, by Books theorem, G is either complete or an odd cycle. In both cases, $\chi_s(G) = \triangle(G) + 1$ by Proposition 7.1. If $\chi(G) -1 \leq \triangle(G)$, *then*, $\chi(G) -1 \leq \triangle(G) \leq \chi_s(G)$. This completes the proof. Note that the result is not true when $G = K_2$, since $\chi(K_2) = 2$. We now obtain tight bounds for two classes of graphs.

Proposition 7.4: If G is a block graph or a cactus, then, semi – strong chromatic number of a graph

$\triangle \leq \chi_2(G) \leq \triangle + 1$. (2)

Proof: In view of (1), it suffices to establish the upper bound. We do this by describing an s-coloring of G with △ + 1 colors. Let G be a block graph. If G exactly one block, then G is complete, and $(G) = \triangle + 1$ if $G \neq K_2$. if $G \neq K_2$, then, $\chi_s(G) = \triangle$. Suppose G has more than one block. Let v be a cut vertex with deg v =△. First color v and its neighbors differently with △ + 1 color. Next, consider a cut vertex u adjacent to v in a block B. All the neighbors of u in B being already colored, color the other neighbors of u differently with colors not used in coloring the vertices of B. This is possible since there are △+1 colors and deg $u \leq \triangle$.Continuing this process, we can color the entire graph G with △ + 1 colors. In the above coloring we observe that any two vertices adjacent to a given vertex are colored differently, and thus this is an s- coloring of G with △ + 1 colors. Now let G be a cactus. We prove $\chi_2(G) \leq \triangle + 1$ by induction on the number n of blocks in G. If $n=1$, the result is true since G is either K_2 *or* a cycle. Suppose $n > 1$, and the result is true for all cacti with n blocks. Let G be a cactus with n +1 blocks and let B be an end block of G with cut vertex v. Remove the vertices of B except v from G resulting in a cactus F. With n blocks. By hypothesis, F can be s-colored with $\triangle(F) + 1$ (and hence with $\triangle(G)$ +1) colors. Consider such a coloring of F with $\triangle(G)$ +1 color. If B is a cycle, then $\deg_F v + 2 = \deg_G v \leq \triangle$. Therefore, out of $\triangle(G)$ +1 colors, at most $\triangle(G)$ -1 colors have been used to color v and its neighbors in F Using the remaining two colors, and the color of v, the cycle B can be colored such that no two neighbours of a vertex are colored the same. This gives an S – coloring of G with $\triangle(G)$ +1colors.If $B = K_2$, a similar argument holds. This completes the proof. We now characterize block graphs for which the upper and lower bounds in (2) are attained.

Proposition 7.5: If G is a block graph, then $\chi_2(G) \le \triangle + 1$ if and only if there exists a vertex of maximum degree which does not belong to a K_2 - block.

Proof: Let deg v = $\triangle$ in *G*. We observe that f no block at v is K_2, the vertex in $N[v]$ must be assigned distinct colors in any *s*-coloring of *G*. This implies $\chi_2(G) \le \triangle + 1$, and hence by (2), $\chi_2(G) \le \triangle + 1$. Conversely, suppose at every vertex of maximum degree in *G*, K_2 is a block. We now describe an *s*-coloring of *G* with $\triangle$ colors, and then by (2), this implies $\chi_2(G) = \triangle$. Let deg v $=\triangle$, and let *uv* be a block K_2 at *v*. First color *v* and its neighbours with $\triangle$ colors giving the same color to v and u, and different colors to all other neighbours of v. Next, let v_1 be a cut vertex adjacent to v. If deg $v_1=\triangle$, there exists a block K_2 at v_1, and an *s*-coloring of v_1 and its neighbours with $\triangle$ colors as above. If deg $v_1< \triangle$, the vertices in $N[v]$ can be colored differently using at most $\triangle$ colors, which turns out to be an *s*-coloring of v_1 and its neighbours. Continuing this process, we have an s-coloring of *G* with $\triangle$ colors. This implies $\chi_2(G) \le \triangle$, and by (2), $\chi_2(G) = \triangle$. This completes the proof.
We note that the lower bound in (2) is attained for a cactus *C* when $C = C_n$, n=0 mod 4. The upper bound is attained for a cactus *C* when $C = C_n$, n=0 mod 4. Also, one can rove that if no vertex *v* of the cactus *G* with v = $\triangle$ belongs to a K_3, then at $\chi_2(G) = \triangle$.

Proposition 7.6: If G a K_3, - free graph of order *p*, and has no isolated vertices then, $\chi_2(G) \le \alpha_1$.

Proof: Let $u_i v_i$, $1 \le i \le \beta_1$ be the edges of matching in G, and $F_i = \{u_i v_i\}$.

Then, $\{F_1, F_2, ..., F_{\beta_1}\}$ together with $p - 2\beta_1$ singleton subsets of $V(G) - \bigcup_{i=1}^{\beta_1} F_i$ from an s- partition of G, since G is K_3–free. Hence $\chi_2(G) \le (p - 2\beta_1) + \beta_1 = p - 2\beta_1 = \alpha_1$.

Proposition 7.7: Let G be a K_3–free graph of order p with a 1-factor. Then, $\chi_2(G) \le \frac{p}{2}$.

Proof: By Proposition 7.6, we have $\chi_2(G) \le \alpha_1 = p - \beta_1 = \frac{p}{2}$.

Proposition 7.8: Let G be a graph of order *p* with diameter 2. Then, $\chi_2(G) \ge [\frac{p}{2}]$.

Proof: Let $\{V_1, V_2, ..., V_k\}$be an s-partition of G. Then, any two vertices u and v in any, V_i, $1 \le i \le k$ should be adjacent. For otherwise, since diameter of G is 2, there exists a vertex w adjacent to both u and v, which is not true. Since any vertex in V_i can have at most one of its neighbors in V_i, each sub graph $\langle V_i \rangle$ is either K_1 or K_2, and the result follows. From Proposition 7.7 and Proposition 7.8, we can deduce the following proposition.

Proposition7.9: If G is a K_3-free graph of order *P* with diameter 2 and a 1-factor, then $\chi_s(G) = \frac{p}{2}$.

Proof: There exist several graphs satisfying the conditions of Proposition 7.7 For example, let G be the graph obtained from C_8 by joining every pair of vertices Vi and V_j with $d(v_i, v_j)$ -= 4. Also, let H be the graph obtained from C_{10} by joining every pair of vertices v_i and v_i with $d(v_i, v_j) = 3$ or 5. Then, both G and H are K_3-free, have diameter 2 and a 1-factor. For both these graphs as well as for the Petersen graph $\chi_s = \frac{p}{2}$, by Proposition 7.7 We now characterize graphs G for which $\chi_s(G) = |V(G)|$.Let $\gamma(G)$ and d(G) respectively denote the domination number and the diameter of a graph G.
The following proposition is due to Brigham and Dutton.

Proposition 7.10: The following are equivalent for graphs G on $p \ge 3$ vertices:

(1) $N(G) = K_p$
(2) $d(G) \le 2$ and every edge of G is on a triangle.
(3) $\gamma(G) \ge 3$, where G is the complement of G.

Proposition 7.11: The following are equivalent for graphs G on $p \ge 3$ vertices:

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(a) χ_s (G) = p.
(b) d(G) ≤ 2 and every edge of G is on a triangle.
(c) y (G) ≥ 3.

Proof: (a) ⇒ (b). χ_s (G) -= p ⇒ χ (N(G)) = p ⇒ N(G) = Kp and this implies (b) by **Proposition 7.10.** (b) ⇒ (c) and (c) ⇒ (a) by **Proposition 7.10**. This completes the proof.
Several graphs exist for which **Proposition 7.11** holds. For example, in C_{2n-1} where n is odd, join all pairs of distinct vertices v_i and v_j if $d(v_i,v_j) \le \frac{(n-1)}{2}$. Let G be the graph thus obtained. Similarly, let H be the graph obtained from C_{2n} (n even) by joining all pairs of distinct vertices vi and v_j if $d(v_i,v_j) \le \frac{n}{2}$. Then, χ_s (G) = 2n-1, and χ_s (H) = 2n.Let ω(G) denote the order of a maximum clique in G. The following result provides a sufficient condition for χ_s (G) = χ_s ($\bar{G}$).The next proposition is due to Brigham and Dutton.

Proposition 7.12: Let G be a graph of order at least three. Then, any two of the following properties implies the third.
(1) N(G) = $\bar{G}$
(2) ω(G) = 2
(3) d(G) = 2.

Proposition 7.13: Let G be a K_3 – free graph with diameter two. Then, χ_s (G) = χ_s ($\bar{G}$).

Proof: This follows from **Proposition 7.12** since the given conditions imply N(G) = G.Clearly, the sub graph induced by the union of any two sets in a χ(G)-partition of G is bipartite. This is also true for any χ_s(G)–partition of G. This completes the proof.

Proposition 7.14: Let L = { V_1, V_2,…. V_k} be a χ_s (G) – partition of G. The sub graph H_{ij}= ⟨ V_i U V_j⟩ of G is bipartite for all i and j.

Proof: Each component in the sub graphs H_i + ⟨ V_i ⟩ is either K_1 or K_2. Also, each vertex in V_i has at most one neighbour in V_j and vice versa. Hence, the degree of any vertex in H_{ij} is at most two. Suppose the sub graph H_{ij} contains a cycle. Since a vertex of degree zero in H_i or Hj has degree at most one in H_{ij}, the vertices o the cycle have degree one in H_i or H_j. Hence, the edges of the cycle are the edges in H_i or H_j, or edges from V_i to V_j. Thus, if e_1,e_2….,e_n are the edges of the cycle, and $e_1 \in H_i$, say, then, e_2 is an edge from Vi to V_j $e_3 \in$ H_j, e_4 is an edge from V_j to V_i etc. This implies n = o mod 4, and proves that every component in H_{ij} is either a path or a cycle C_n, n= o mod 4. H_{ij} is bipartite.

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